Upper bound on the heat transport in a layer of fluid of infinite Prandtl number, rigid lower boundary, and stress-free upper boundary

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We obtain an upper bound on the convective heat transport in a heated from below horizontal fluid layer of infinite Prandtl number with rigid lower boundary and stress-free upper boundary. Because of the asymmetric boundary conditions the solutions of the Euler-Lagrange equations of the corresponding variational problem are also asymmetric with different thicknesses of the boundary layers on the upper and lower boundary of the fluid. The obtained bound on the convective heat transport and the corresponding wave number are between the values for a fluid layer with two rigid boundaries and a fluid layer with two stress-free boundaries.

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The methods of the optimum theory of turbulence allow us to obtain rigorous results for the turbulent transport quantities in fluid systems directly on the basis of the governing Navier-Stokes equations. Following earlier ideas of Malkus [1,2], Howard [3] formulated the theory of the upper bounds for the convective heat transport in a horizontal fluid layer, heated from below. Busse [4] contributed to the optimum theory of turbulence by the introduction of the multiwave number solutions of the arising variational problems. In the following years the Howard-Busse method was applied for obtaining bounds on turbulent quantities in different cases of flows and thermal convection [5–7]. In 1992 Doering and Constantin [8] have proposed another method for deriving bounds on turbulent quantities. This method is based on the background flow idea. Its strong side is that if the background field satisfies certain spectral constraint one obtains immediately an upper bound on the investigated turbulent quantity [9–11]. Up to now the results obtained by the Howard-Busse method persist the attempts for further improvement. There exist indications that the variational problems of Howard and Busse and Doering and Constantin may be equivalent. Progress in this area of investigation for the case of shear flow has been made recently by Kerswell [12].

In this paper we apply the Howard-Busse method and obtain an upper bound on the heat transport in a horizontally infinite layer of fluid with rigid lower boundary and stress-free upper boundary. The cases of a fluid layer with two rigid boundaries and two stress-free boundaries are discussed in Refs. [13] and [14]. For these cases the optimum fields are symmetric with respect to the midplane of the fluid layer. For the case, discussed below, the boundary conditions are asymmetric. This leads to asymmetric layer structure of the optimum fields as well as to different values for the upper bound on the convective heat transport, wave number, and the thicknesses of the boundary layers of the optimum fields.

Consider a horizontally infinite fluid layer heated from below of thickness d with fixed temperatures T_1 and T_2 at the upper and lower boundaries. Let us denote the coefficient of thermal expansion by γ, the kinematic viscosity by ν, and the acceleration of gravity by g. Denoting the thermal diffusivity of the fluid as κ and using d as length scale, d^2/κ as time scale, and $(T_2-T_1)/R$ as temperature scale we can write the Navier-Stokes equations for the velocity vector $\mathbf{u}$ and the heat equation for the deviation Θ from the static temperature distribution in dimensionless form

$$(1/P)(\partial \mathbf{u}/\partial t+\mathbf{u}\cdot\boldsymbol{\nabla}\mathbf{u})=-\nabla p+\mathbf{k}\Theta+\nabla^2\mathbf{u}, \tag{1}$$

$$\nabla\cdot\mathbf{u}=0, \tag{2}$$

$$\partial\Theta/\partial t+\mathbf{u}\cdot\nabla\Theta=R\mathbf{k}\cdot\mathbf{u}+\nabla^2\Theta, \tag{3}$$

where we have introduced a Cartesian system of coordinates with z axis in the vertical direction. $R=\gamma(T_2-T_1)gd^3/(\kappa\nu)$ is the Rayleigh number, $P=\nu/\kappa$ is the Prandtl number, and $\mathbf{k}$ is the vertical unit vector. Denoting the z component of $\mathbf{u}$ as w we write for the rigid boundary conditions on the bottom of the layer $w=\partial w/\partial z=\Theta=0$ at $z=-1/2$ and for the stress-free boundary conditions on the top of the layer $w=\partial^2 w/\partial z^2=\Theta=0$ at $z=1/2$.

We introduce the averages of a quality q over the the planes $z=\text{const}$ (denoted as $\overline{q}$) and over the fluid layer (denoted as $\langle q\rangle$) (for definitions see Ref. [3]). The temperature field is separated into two parts $\Theta=\overline{\Theta}+T$ such that $\overline{T}=0$ holds. A subtraction of the horizontal average of Eq. (3) from Eq. (3) leads us to the result

$$\frac{\partial T}{\partial t}+\mathbf{u}\cdot\nabla T+w\frac{\partial\overline{\Theta}}{\partial z}-\frac{\partial}{\partial z}\overline{wT}=Rw+\nabla^2 T. \tag{4}$$

Multiplying Eq. (1) by $\mathbf{u}$ and Eq. (4) by T and averaging the result over the fluid layer we obtain the relationships

$$(1/2P)d\langle\mathbf{u}\cdot\mathbf{u}\rangle/dt=\langle wT\rangle-\langle|\nabla\mathbf{u}|^2\rangle, \tag{5}$$

$$(1/2)d\langle T^2\rangle/dt=R\langle wT\rangle-\langle|\nabla T|^2\rangle-\langle\overline{wT\partial\Theta}/\partial z\rangle. \tag{6}$$

We are interested in turbulent convection long after any external parameter has been changed. We define this situation by the condition that all horizontally averaged quantities are time independent. In this case we obtain from the first integral of the horizontal averaged (3): $d\overline{\Theta}/dz=\overline{wT}-\langle wT\rangle$. Using this we obtain the final form of the relationships (5),(6) (known also as power integrals)

$$\langle wT\rangle=\langle|\nabla\mathbf{u}|^2\rangle, \tag{7}$$

$$\langle|\nabla T|^2\rangle=R\langle wT\rangle+\langle wT\rangle^2-\langle\overline{wT^2}\rangle. \tag{8}$$

Equations (7),(8) hold for all Prandtl numbers P. The imposition of the infinite Prandtl number condition allows a further restriction of the fields that satisfy the power integrals. Equation (1) becomes linear in the limit $P\rightarrow\infty$ and we shall incorporate it as an additional constraint into the variational problem. The pressure is eliminated by taking the z-component of the double curl of Eq. (1). Thus we obtain $\nabla_1^2T+\nabla^4w=0$.

We take also the equation of continuity as a constraint into the variational problem by means of the general representation of a solenoidal vector field $\mathbf{u}$ in terms of a poloidal and a toroidal component $\mathbf{u}=\nabla\times(\nabla\times\mathbf{k}\phi)+\nabla\times\mathbf{k}\psi$ where the condition $\overline{\phi}=\overline{\psi}=0$ can be imposed without changing $\mathbf{u}$. Taking the curl of Eq. (1) we see that ψ must vanish in the limit of infinite Prandtl number. The z component of $\mathbf{u}$ of interest to us is given by the poloidal field ϕ, $w=-\nabla_1^2\phi$ where $\nabla_1=\partial^2/\partial x^2+\partial^2/\partial y^2$.

We write a functional for the convective heat transport $Nu-1=\langle wT\rangle/R$ where Nu is the Nusselt number. Using the power integrals (7) and (8) and imposing the normalization condition $\langle w\theta\rangle=1$ we obtain the variational problem in the following form.

Given Rayleigh number R find the maximum $F(R)$ of the variational functional

$$\mathcal{F}(R,w,\theta)=[1-(1/R)\langle|\nabla\theta|^2\rangle]/\langle(1-\overline{w\theta})^2\rangle \tag{9}$$

among all fields w and θ subject to the constraints $\langle w\theta\rangle=1$ and $\nabla^4w+\nabla_1^2\theta=0$ and the boundary conditions $w=\partial w/\partial z=\theta=0$ at $z=-1/2$ and $w=\partial^2w/\partial z^2=\theta=0$ at $z=1/2$. The Euler-Lagrange equations for the functional (9) are

$$\frac{1}{RF}\nabla^6\theta+\nabla^4\left[\left(1-\overline{w\theta}-\frac{2\lambda}{F}\right)w\right]+(1-\overline{w\theta})\nabla^4w=0, \tag{10}$$

$$\nabla^4w+\nabla_1^2\theta=0, \tag{11}$$

where λ is $-1\leqslant\lambda=-(1/2)[2-(1/R)\langle|\theta|^2\rangle]\leqslant-1/2$.

We assume that five sublayers of the fluid layer exist. At high Rayleigh numbers almost the whole layer volume is occupied by an internal layer in which the optimum fields w and θ are almost constants and $\overline{w\theta}$ is also constant approximately equal to 1. On the borders of this internal layer we have two intermediate layers in which $\overline{w\theta}$ is again almost constant and equal to 1 but w and θ change substantially. A boundary layer develops between each of the intermediate layers and the boundary of the fluid layer. In this layer the fields w and θ satisfy the boundary conditions and match the correspondent values on the border between the boundary and the intermediate layer.

For each sublayer we introduce the correspondent coordinates. For the internal layer the coordinate remains z. For the two intermediate sublayers we introduce the coordinates $\xi_l=\alpha(z+1/2)$ for the lower intermediate layer and $\xi_u=\alpha(1/2-z)$ for the upper intermediate layer. For the boundary layers we introduce the coordinates $\eta_l=(\alpha/\delta_l)(z+1/2)$ for the lower boundary layer (near the rigid boundary) and $\eta_u=(\alpha/\delta_u)(1/2-z)$ for the upper boundary layer (near the stress-free boundary). δ_u and δ_l are connected to the thickness of the correspondent boundary layer and have the property $\delta_{u,l}\rightarrow0$ when $R\rightarrow\infty$.

In the internal layer the assumption is $\overline{w\theta}\approx1$ and $w_1\approx$const, $\theta_1\approx$const which lead to vanishing of the terms containing derivatives. Thus we obtain the solution $w_1=\tilde{w}_1/\alpha;\theta_1=\tilde{\theta}_1\alpha;\ \tilde{w}_1=\tilde{\theta}_1=1$. For the intermediate layers we write w_1 and θ_1 as $w_1=\check{w}_1(\xi_u)/\alpha;\ \theta_1=\alpha\check{\theta}_1(\xi_u);\ w_1=\check{w}_1(\xi_l)/\alpha;\ \theta_1=\alpha\check{\theta}_1(\xi_l)$. Introducing the operator $\hat{L}_{l,u}=(d^2/d\xi_{l,u}^2-1)$ we obtain the resulting Euler-Lagrange equations for the upper and lower intermediate layers

$$\frac{\alpha^4}{RF}\hat{L}_{l,u}^3\check{\theta}_1+\hat{L}_{l,u}^2[(1-\check{w}_1\check{\theta}_1)\check{w}_1]+(1-\check{w}_1\check{\theta}_1)\hat{L}_{l,u}^2\check{w}_1=0, \tag{12}$$

$$\hat{L}_{l,u}^2\check{w}_1=\check{\theta}_1. \tag{13}$$

The solution of Eq. (12) is approximately $\check{w}_1\check{\theta}_1=1$. The approximate solution of Eq. (13) when $\xi_l\rightarrow0$ is $\check{w}_1=\xi_l^2\sqrt{\ln(1/\xi_l)}$. When $\xi_u\rightarrow0$ the correspondent solution is $\check{w}_1=c\xi_u-[\xi_u^3/(6c)]\ln(1/\xi_u)$ with $c=0.834210$.

We shall match the above solutions to the solutions for the boundary layers. For the upper and for the lower boundary layers we perform the scaling $w_1=A_u\hat{w}_1(\eta_u);\ \theta_1=(1/A_u)\hat{\theta}_1(\eta_u);\ w_1=A_l\hat{w}_1(\eta_l);\ \theta_1=(1/A_l)\hat{\theta}_1(\eta_l)$. Under the assumption that in the boundary layers the terms containing the higher derivatives are much larger than the other terms we obtain for the Euler-Lagrange equations in the boundary layers

$$(\alpha^2A_{u,l}^2/\delta_{u,l}^4)(d^4\hat{w}_1/d\eta_{u,l}^4)=\hat{\theta}_1, \tag{14}$$

$$\frac{\alpha^2}{A_{u,l}^2RF\delta_{u,l}^2}\frac{d^6\hat{\theta}_1}{d\eta_{u,l}^6}+\frac{d^4}{d\eta_{u,l}^4}[(1-\hat{w}_1\hat{\theta}_1)\hat{w}_1]+(1-\hat{w}_1\hat{\theta}_1)\frac{d^4\hat{w}_1}{d\eta_{u,l}^4}=0. \tag{15}$$

The boundary conditions for the upper boundary layer are $\hat{w}_1(0)=\hat{w}_1''(0)=\hat{\theta}_1(0)=0$ and the boundary conditions for the lower boundary layer are $\hat{w}_1(0)=\hat{w}_1'(0)=\hat{\theta}_1(0)=0$.

The solution for the upper intermediate layer must match the solution for the upper boundary layer. Using that $\xi_u=\delta_u\eta_u$ and assuming $A_u=c\delta_u/\alpha$ we obtain the approximate solution for the upper boundary layer $\hat{w}_1\approx\eta_u+\cdots$. Analogous the solution for the lower intermediate layer must match the solution in the lower boundary layer. Using that $\xi_l=\delta_l\eta_l$ and assuming $A_l=(1/\alpha)\delta_l^2[\ln(1/\delta_l)]^{1/2}$ we obtain the approximate solution for the lower boundary layer $\hat{w}_1\approx\eta_l^2+\cdots$

For F, α, δ_u, δ_l we have the equations

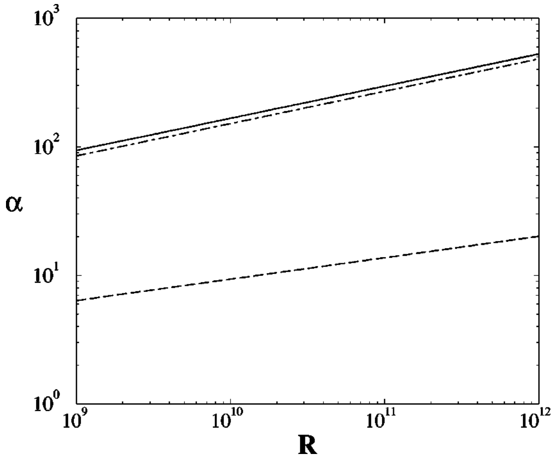

FIG. 1. Wave number α as function of the Rayleigh number. Solid line: case of fluid layer with two rigid boundaries. Dashed line: case of fluid layer with two stress-free boundaries. Dot-dashed line: case of fluid layer with rigid lower boundary and stress-free upper boundary.

$$\alpha^4 = c^2 \delta_u^4 RF, \quad \alpha^4 = RF\delta_l^6 \ln(1/\delta_l), \tag{16}$$

$$\partial F/\partial \alpha = 0, \tag{17}$$

$$F = [2\alpha(1-\alpha^4/R)]/[\delta_u D_u + \delta_l D_l], \tag{18}$$

where

$$D_u = 2\int_0^\infty d\eta_u \left[\left(\frac{d\hat{\theta}_1^{(1)}}{d\eta_u} \right)^2 + (1-\eta_u \hat{\theta}_1^{(1)})^2 \right] = 2.11975, \tag{19}$$

$$D_l = 2\int_0^\infty d\eta_l \left[\left(\frac{d\hat{\theta}_1^{(1)}}{d\eta_l} \right)^2 + (1-\eta_l^2 \hat{\theta}_1^{(1)})^2 \right] = 2.2212. \tag{20}$$

An application of Eq. (17) to Eq. (18) leads us to the relationship

$$\delta_u = \delta_l (D_l/D_u)[R - 13\alpha^4]/[12\alpha^4]. \tag{21}$$

The solution for δ_l is

$$\delta_l = A^*(R)^{-1} \left(\frac{13 D_l^4}{24^4} \right)^{1/20} \left(\frac{1}{20} \right)^{-1/5} R^{-1/20} (\ln R)^{-1/5}, \tag{22}$$

where $A_1 = 24^4 c^{1/2}/(12 D_u D_l^3)$; $A_2 = (13 D_l^4/24^4)^{-39/40} (1/20)^{3/20} R^{-1/40} (\ln R)^{3/20}$; and $A^*(R) \approx [A_1/(A_1+A_2)]^{1/20}$. The result for α is

$$\alpha = [A^*(R)]^5 (R/13)^{1/4} \tag{23}$$

and the result for F is

$$F = \left(\frac{24}{D_l} \right)^{6/5} [A^*(R)]^{26} 13^{-13/10} 20^{-1/5} R^{3/10} (\ln R)^{1/5}. \tag{24}$$

Figures 1–3 show the dependencies $F(R)$, $\alpha(R)$, $\delta_u(R)$, $\delta_l(R)$ in comparison to the correspondent functions obtained for the cases of fluid layer with two stress-free or two rigid boundaries. As it can be expected the upper bound on the convective heat transport for the case discussed here lies between the bounds on the convective heat transport for the cases of layer with two rigid boundaries and of layer with two stress-free boundaries. The wave number α is close to the wave number for the case of two rigid boundaries. The contribution of the upper boundary and intermediate layers is expressed by the term $A^*(R)$. The asymmetry of the optimum fields leads to different dependencies of the thickness of the upper and lower boundary layer on the Rayleigh number. Instead of one equation for the boundary layer thickness δ for the case of a fluid layer with two rigid or two stress-free boundaries, here we have two equations for the boundary layer thicknesses δ_u, δ_l.

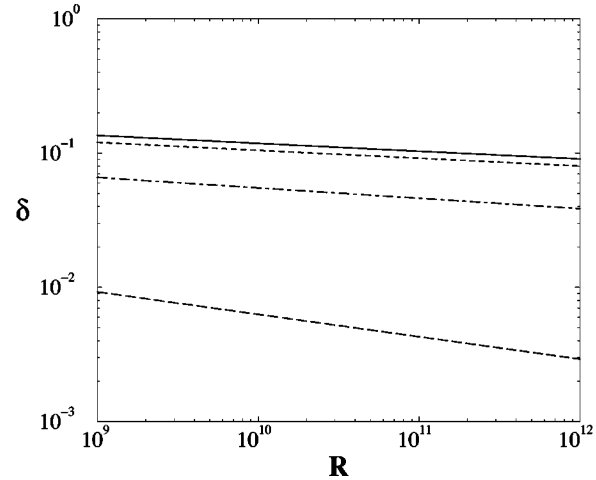

FIG. 2. Boundary-layer thicknesses of the optimum fields as function of the Rayleigh number. Solid line: case of the fluid layer with two rigid boundaries. Dashed line: case of fluid layer with two stress-free boundaries. Dot-dashed line: upper boundary layer for the case of a fluid layer with rigid lower boundary and stress-free upper boundary. Short-dashed line: lower boundary layer for the case of fluid layer with rigid lower and stress-free upper boundary.

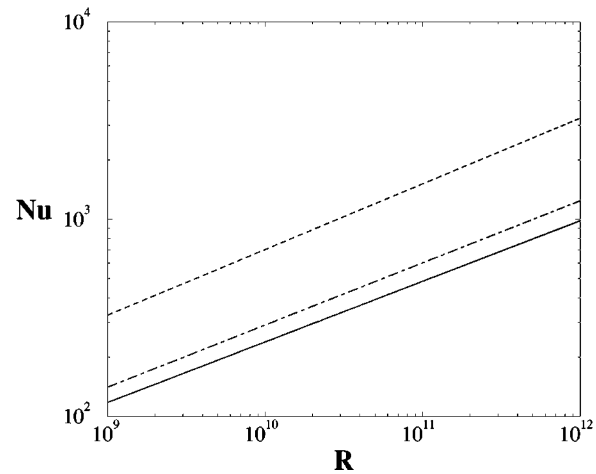

FIG. 3. Upper bound on the convective heat transport as function of the Rayleigh number. Solid line: case of fluid layer with two rigid boundaries. Dashed line: case of fluid layer with two stress-free boundaries. Dot-dashed line: case of fluid layer with rigid lower boundary and stress-free upper boundary.

We note that the theory of the $1-\alpha$ solution of the variational problem contains as a particular case the problem for the onset of the convection. Analogous to Ref. [15] the functional, obtained by the power integrals (7), (8) can be reduced to the variational functional that determines the onset of the convection (for details see Ref. [16]) and thus we obtain for the critical values of the wave-number and Rayleigh number: $\alpha_c = 2.682$, $R_c = 1100.65$.

The asymptotic theory of the $1-\alpha$ solution of the variational problem outlines the routes which must be followed when a theory of the multi-α solutions is developed. The boundary layers are important part of the theory of the multi-wave-numbers solutions. The increasing of the number of the boundary layers which must be taken into account leads to complications in the multi-α case. The equations we shall have to solve will be more complicated than these for the case of a fluid layer with two rigid boundaries [13]. The investigation reported here indicates that despite the complications because of the asymmetry of the optimum fields we could obtain asymptotic analytical upper bounds on the convective heat transport based on the multi-α solutions of the variational problem. It is well known that the $1-\alpha$ solution of the variational problem leads to rigorous upper bound on the convective heat transport up to certain value of the Rayleigh number. If the Rayleigh number increases further the rigorous upper bound is obtained on the basis of the $2-\alpha$ solution, then by the $3-\alpha$ solution, etc. As the upper bound obtained by the $1-\alpha$ solution for the case, discussed here, is between the bounds for the cases of fluid layer with two rigid and two stress-free boundaries, we can expect, that the bound obtained by the $N-\alpha$ solution of the variational problem ($N \to \infty$) will be $\propto R^{1/3}$. Indications for such a result can also be seen from the inequality obtained in Ref. [17]. The theory of the multi-α solutions of the variational problem which will answer the questions about the regions of the validity of the bounds obtained by each of the multi-α solutions of the variational problem, about the value of the bound obtained when the number of the wave numbers tends to infinity, and about the relations among the thicknesses of the upper and lower boundary layers of the optimum fields, will be a subject of future research.

[1] W.V.R. Malkus, Proc. R. Soc. London, Ser. A **225**, 185 (1954).

[2] W.V.R. Malkus, Proc. R. Soc. London, Ser. A **225**, 196 (1954).

[3] L.N. Howard, J. Fluid Mech. **17**, 405 (1963).

[4] F.H. Busse, J. Fluid Mech. **37**, 457 (1969).

[5] L.N. Howard, Annu. Rev. Fluid Mech. **4**, 473 (1972).

[6] F.H. Busse, Adv. Appl. Mech. **18**, 77 (1978).

[7] F.H. Busse, *Energy Stability and Convection*, edited by G.P. Galdi and B. Straughan (Pitman, Boston, 1988).

[8] C.R. Doering and P. Constantin, Phys. Rev. Lett. **69**, 1648 (1992).

[9] C.R. Doering and P. Constantin, Phys. Rev. E **53**, 5957 (1996).

[10] C.R. Doering and J.M. Hyman, Phys. Rev. E **55**, 7775 (1997).

[11] N.P. Hoffmann, and N.K. Vitanov, Phys. Lett. A **255**, 277 (1999).

[12] R.R. Kerswell, Physica D **121**, 175 (1998).

[13] S.-K. Chan, Stud. Appl. Math. **50**, 13 (1971).

[14] N.K. Vitanov, Phys. Lett. A **248**, 338 (1998).

[15] N.K. Vitanov and F.H. Busse, J. Appl. Math. Phys. **48**, 310 (1997).

[16] S. Chandrasekhar, *Hydrodynamic and Hydromagnetic Stability* (Dover, New York, 1981).

[17] P. Constantin and C.R. Doering, J. Stat. Phys. **94**, 159 (1999).